INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

NOTE

- This paper will be followed by **Paper B: Comprehension**.

Section A (36 marks)

1 Express $\dfrac{x}{x^2-4}+\dfrac{2}{x+2}$ as a single fraction, simplifying your answer. [3]
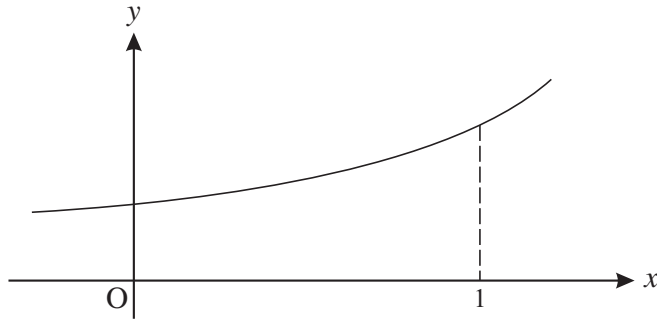
2 Fig. 2 shows the curve $y=\sqrt{1+e^{2x}}$.

Fig. 2

The region bounded by the curve, the x-axis, the y-axis and the line $x=1$ is rotated through $360°$ about the x-axis.

Show that the volume of the solid of revolution produced is $\frac{1}{2}\pi(1+e^2)$. [4]

3 Solve the equation $\cos 2\theta = \sin\theta$ for $0 \leqslant \theta \leqslant 2\pi$, giving your answers in terms of π. [7]

4 Given that $x=2\sec\theta$ and $y=3\tan\theta$, show that $\dfrac{x^2}{4}-\dfrac{y^2}{9}=1$. [3]

5 A curve has parametric equations $x=1+u^2$, $y=2u^3$.

   (i) Find $\dfrac{dy}{dx}$ in terms of u. [3]

   (ii) Hence find the gradient of the curve at the point with coordinates (5, 16). [2]

6 (i) Find the first three non-zero terms of the binomial series expansion of $\dfrac{1}{\sqrt{1+4x^2}}$, and state the set of values of x for which the expansion is valid. [5]

   (ii) Hence find the first three non-zero terms of the series expansion of $\dfrac{1-x^2}{\sqrt{1+4x^2}}$. [3]

7 Express $\sqrt{3}\sin x - \cos x$ in the form $R\sin(x-\alpha)$, where $R>0$ and $0<\alpha<\frac{1}{2}\pi$. Express α in the form $k\pi$.

Find the exact coordinates of the maximum point of the curve $y=\sqrt{3}\sin x-\cos x$ for which $0<x<2\pi$. [6]

Section B (36 marks)

8   The upper and lower surfaces of a coal seam are modelled as planes ABC and DEF, as shown in Fig. 8. All dimensions are metres.
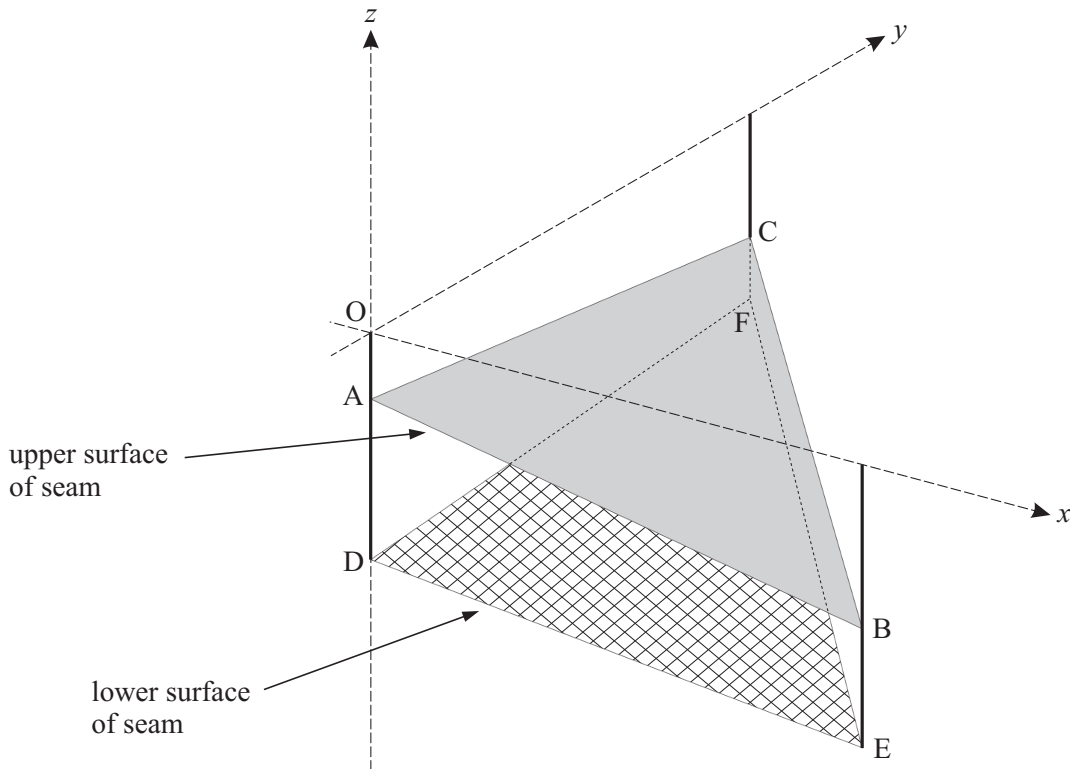

Fig. 8

Relative to axes Ox (due east), Oy (due north) and Oz (vertically upwards), the coordinates of the points are as follows.

A: (0, 0, −15)   B: (100, 0, −30)   C: (0, 100, −25)
D: (0, 0, −40)   E: (100, 0, −50)   F: (0, 100, −35)

(i) Verify that the cartesian equation of the plane ABC is $3x + 2y + 20z + 300 = 0$.   [3]

(ii) Find the vectors $\overrightarrow{DE}$ and $\overrightarrow{DF}$. Show that the vector $2\mathbf{i} - \mathbf{j} + 20\mathbf{k}$ is perpendicular to each of these vectors. Hence find the cartesian equation of the plane DEF.   [6]

(iii) By calculating the angle between their normal vectors, find the angle between the planes ABC and DEF.   [4]

It is decided to drill down to the seam from a point R (15, 34, 0) in a line perpendicular to the upper surface of the seam. This line meets the plane ABC at the point S.

(iv) Write down a vector equation of the line RS.

Calculate the coordinates of S.   [5]

9 A skydiver drops from a helicopter. Before she opens her parachute, her speed $v\,\mathrm{m\,s}^{-1}$ after time t seconds is modelled by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10\mathrm{e}^{-\frac{1}{2}t}.$$

When $t = 0$, $v = 0$.

  (i) Find v in terms of t. [4]

  (ii) According to this model, what is the speed of the skydiver in the long term? [2]

She opens her parachute when her speed is $10\,\mathrm{m\,s}^{-1}$. Her speed t seconds after this is $w\,\mathrm{m\,s}^{-1}$, and is modelled by the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\tfrac{1}{2}(w-4)(w+5).$$

(iii) Express $\dfrac{1}{(w-4)(w+5)}$ in partial fractions. [4]

(iv) Using this result, show that $\dfrac{w-4}{w+5} = 0.4\mathrm{e}^{-4.5t}$. [6]

  (v) According to this model, what is the speed of the skydiver in the long term? [2]

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RECOGNISING ACHIEVEMENT

ADVANCED GCE   **4754/01B**

MATHEMATICS (MEI)

Applications of Advanced Mathematics (C4) Paper B: Comprehension

WEDNESDAY 21 MAY 2008   Afternoon

Time: Up to 1 hour

Additional materials: Rough paper
                                 MEI Examination Formulae and Tables (MF2)

Candidate Forename

Candidate Surname

Centre Number

Candidate Number

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the boxes above.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Write your answers in the spaces provided on the question paper.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The insert contains the text for use with the questions.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with your question paper.
- The total number of marks for this paper is 18.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

FOR EXAMINER'S USE	
1	
2	
3	
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5	
6	
7	
TOTAL	

This document consists of **6** printed pages, **2** blank pages and an insert.

          OCR is an exempt Charity   **[Turn over**

1 Complete these Latin square puzzles.

(i)

2	1	3
3		

[1]

(ii)

2	3	
	2	

[1]

2 In line 51, the text says that the Latin square

1	2	3	4
3	1	4	2
2	4	1	3
4	3	2	1

could not be the solution to a Sudoku puzzle.

Explain this briefly. [2]

....

....

....

....

3 On lines 114 and 115 the text says "It turns out that there are 16 different ways of filling in the remaining cells while keeping to the Sudoku rules. One of these ways is shown in Fig. 10."

Complete the grid below with a solution different from that given in Fig. 10.   [2]

1	2	3	4

4 Lines 154 and 155 of the article read "There are three other embedded Latin squares in Fig. 14; one of them is illustrated in Fig. 16."

Indicate one of the other two embedded Latin squares on this copy of Fig. 14.   [2]

4	2	3	1
		2	4
		4	2
2	4	1	3

5 The number of 9×9 Sudokus is given in line 121.

Without doing any calculations, explain why you would expect 9! to be a factor of this number.
[2]

....

....

....

6   In the table below, *M* represents the maximum number of givens for which a Sudoku puzzle may have no unique solution (Investigation 3 in the article). *s* is the side length of the Sudoku grid and *b* is the side length of its blocks.

Block side length, *b*	Sudoku, $s \times s$	*M*
1	1×1	–
2	4×4	12
3	9×9	
4	16×16	
5		

   (i) Complete the table. [2]

   (ii) Give a formula for *M* in terms of *b*. [2]

   ....

   ....

7 A man is setting a Sudoku puzzle and starts with this solution.

1	2	3	4	5	6	7	8	9
4	5	6	8	9	7	3	1	2
7	8	9	3	1	2	5	6	4
2	3	1	5	6	4	8	9	7
5	6	4	9	7	8	1	2	3
8	9	7	1	2	3	6	4	5
3	1	2	6	4	5	9	7	8
6	4	5	7	8	9	2	3	1
9	7	8	2	3	1	4	5	6

He then removes some of the numbers to give the puzzles in parts **(i)** and **(ii)**. In each case explain briefly, and without trying to solve the puzzle, why it does not have a unique solution.

[2, 2]

(i)

1	2		4		6			9
4			8	9			1	
	8						6	
2		1			4			7
	6	4		7	8	1	2	
8	9			2			4	
	1		6	4		9	7	
6	4		7		9			1
9		8	2		1	4		6

(ii)

1	2	3	4	5	6	7	8	9
4	5	6	8	9	7	3	1	2
7	8	9				5	6	4
2	3	1	5	6	4	8	9	7
5	6	4	9	7	8	1	2	3
8	9	7				6	4	5
3	1	2	6	4	5	9	7	8
6	4	5	7	8	9	2	3	1
9	7	8				4	5	6

(i) ....

....

....

....

(ii) ....

....

....

....

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8

ADVANCED GCE **4754/01B**

MATHEMATICS (MEI)

Applications of Advanced Mathematics (C4) Paper B: Comprehension

INSERT

WEDNESDAY 21 MAY 2008   Afternoon

Time: Up to 1 hour

INSTRUCTIONS TO CANDIDATES

- This insert contains the text for use with the questions.

This document consists of **11** printed pages and **1** blank page.

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Sudoku puzzles

Introduction

Sudoku puzzles were introduced to the UK in 2004 and almost immediately became very popular. Many people found them addictive. A common form of Sudoku puzzle uses a 9×9 grid, as illustrated in Fig. 1.

5

- A square grid consists of small squares or *cells*. In the case of the 9×9 grid, there are 81 cells.

- The square grid is also divided into a number of smaller square grids, or *blocks*. In the case of the 9×9 grid, there are 9 blocks; each of them is a 3×3 grid and so contains 9 cells.

- When the puzzle is completed, each row, each column and each block will contain each of 9 different *symbols* exactly once. The symbols used in a 9×9 Sudoku puzzle are usually the numerals 1, 2, 3, 4, 5, 6, 7, 8 and 9 but they could just as well be letters, for example A, B, C, D, E, F, G, H and I.

10

- The symbols in some of the cells are already provided at the start, and these are called the *givens*.

15

- You have to fill in the symbols in the remaining cells. If the puzzle has been set correctly, there is only one solution; it is unique.

[Warning   Do not spend time in this examination trying to complete this puzzle. You are not expected to solve it.]

								1
					3		2	
		4			9			
							9	
					4	6		
	1	7		8				
				1				7
	3		5					
9						4		

Fig. 1

In this article, the term *Sudoku* is used to describe a correctly completed grid whereas *Sudoku puzzle* is used for the initial situation where there are empty cells to be filled in. A Sudoku puzzle leads

20

to a unique Sudoku, but the converse is not true; any Sudoku can be arrived at from many different possible starting points.

There are several variations on the basic Sudoku puzzle. Some involve grids of different shapes but, in this article, only those with square grids containing square blocks are considered. The size of the grid does not need to be 9×9; another commonly used grid is 16×16 (using the symbols 0 to 9 together with A, B, C, D, E and F), and you can even have 25×25 Sudoku puzzles. A simpler possibility is 4×4. Thus, in this article, the side length of the grid, s, is a square number and $s = b^2$, where b is the side length of each block.

Questions about Sudoku puzzles

This article investigates three questions about 9×9 Sudoku puzzles.

1. In how many different ways can you place the symbols 1 to 9 in the 81 cells of a 9×9 Sudoku grid, subject to the Sudoku rules?

2. What is the smallest number of givens that must be provided if a puzzle is to have a unique solution?

3. What is the largest number of givens that can be provided without the puzzle having a unique solution?

Approach

Two different problem-solving techniques are used in this article.

- The modelling approach is to start by working with a similar but simpler problem, see what you can learn from it and then attack the real problem; in this case the simpler problem is the Latin square puzzle.

- The other approach is to look at the same problem but on a smaller scale; you hope for one or both of two outcomes.

  – The smaller scale case will help you to find an appropriate method.

  – You can build a sequence of results from several smaller cases which will suggest a possible formula for a general result.

Latin squares

In a Latin square, the grid is not divided into smaller blocks as in a Sudoku, but each symbol must appear once in each row and in each column. An example of a 4×4 Latin square is given in Fig. 2.

1	2	3	4
3	1	4	2
2	4	1	3
4	3	2	1

Fig. 2

Notice that this Latin square could not be the solution to a Sudoku puzzle.

All Sudokus are Latin squares, but not all Latin squares are Sudokus. Consequently there are more Latin squares of any size (other than the trivial 1×1 case) than there are Sudokus, and so the number of Latin squares provides an upper bound for the number of Sudokus.

Latin squares have been known for a long time and in medieval times were used for decoration in   55
the Islamic world. They were re-popularised in the late 1700s by Leonhard Euler who set puzzles on them that are often said to be the predecessors of Sudokus. These days, Latin squares are widely used in experimental design in statistics. In this article a puzzle which requires you to complete a Latin square is referred to as a Latin square puzzle.

Investigation 1:   How many 9×9 Sudokus are there?   60

The first investigation involves the number of ways that it is possible to allocate the symbols 1, 2, 3, …, 9 to the 81 cells on a Sudoku grid.

The first approach to the problem is to look at Latin squares of different sizes.

Trivially, there is one possible 1×1 Latin square.

There are two possible 2×2 Latin squares and these are shown in Fig. 3.   65

Fig. 3

What about 3×3 Latin squares? Start by thinking about the top row.

- Choose the top left symbol first. It can be 1 or 2 or 3 so there are 3 possibilities.

- Now choose the top middle symbol; whatever the top left symbol, there are now 2 remaining symbols to choose from for this cell.

- There is only 1 possibility for the last symbol in the row, in the top right cell.   70

- So the number of ways of filling the cells in the top row is $3 \times 2 \times 1 = 3! = 6$.
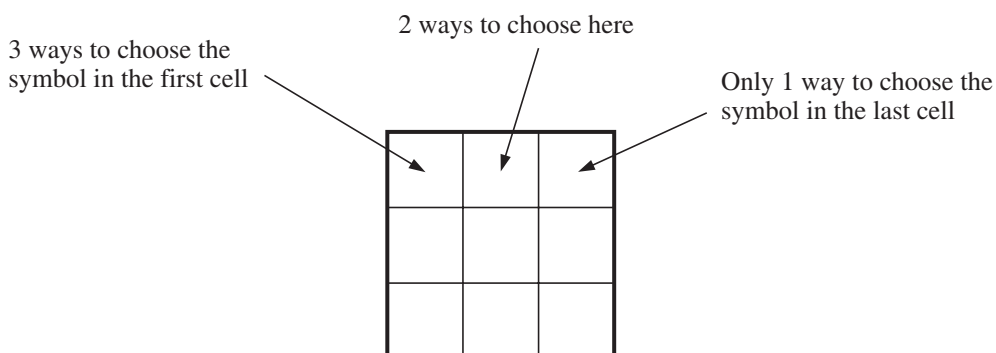

Fig. 4

Now think about the left-hand column.

- The symbol in the top cell has already been chosen.

- There remain 2 possible symbols for the middle left cell.

- There is only 1 possibility for the symbol at the bottom of the column. 75

- So for any given top row, the number of ways of filling the rest of the left-hand column is $2 \times 1 = 2! = 2$.
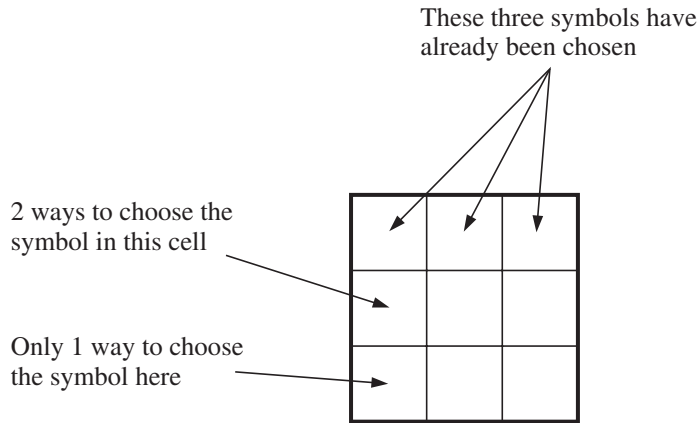
These three symbols have already been chosen

2 ways to choose the symbol in this cell

Only 1 way to choose the symbol here

Fig. 5

In each case there is only one way to fill in the four remaining cells, as shown in the example below where the top row is (1 2 3) and the left-hand column is (1 3 2).
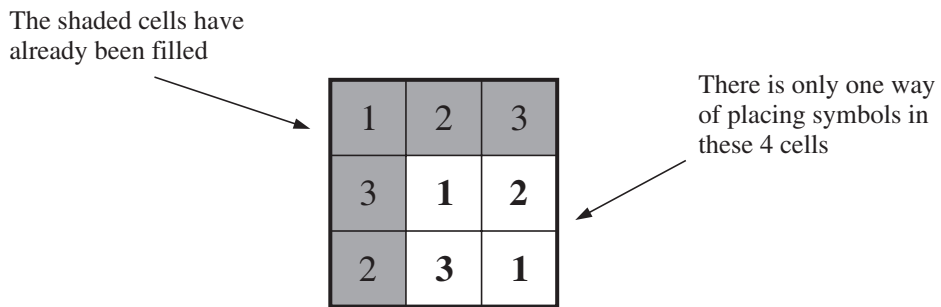
The shaded cells have already been filled

1	2	3
3	**1**	**2**
2	**3**	**1**

There is only one way of placing symbols in these 4 cells

Fig. 6

So the number of 3×3 Latin squares is given by 80

	Top row		Left-hand column		Rest		
	6	$\times$	2	$\times$	1	=	12.
i.e.	3!	$\times$	2!	$\times$	1		

This process is illustrated in Figs. 4, 5 and 6.

The next step is to investigate the 4×4 Latin square, using the same approach.   85

- The number of ways of filling the cells in the top row is $4 \times 3 \times 2 \times 1 = 4! = 24$.

- The top cell in the left-hand column has already been filled, so there are just 3 cells remaining; there are $3 \times 2 \times 1 = 3! = 6$ ways of filling these cells.

- That leaves 9 cells, as shown in the example in Fig. 7. There is nothing special about the symbols in the top row and left-hand column; they are just a typical selection.   90

These 9 cells have still to be filled in

Fig. 7

The next step is to find the number of ways of filling the remaining 9 cells. The answer turns out to be 4.

So the total number of 4×4 Latin squares is given by

$$\begin{array}{ccccccc} \text{Top row} & & \text{Left-hand column} & & \text{Rest} & & \\ 4! & \times & 3! & \times & 4 & = & 576. \end{array}$$ 95

A similar procedure can be used to find the number of 5×5 Latin squares. Considering the top row and the left-hand column gives $5! \times 4!$ but counting the rest is distinctly messy. There are actually 56 ways, so the number of 5×5 Latin squares is $5! \times 4! \times 56 = 161\,280$.

The hope that finding the numbers of Latin squares of sides 1, 2, 3, 4, … would suggest a possible formula now looks rather forlorn.   100

Table 8 gives the numbers of Latin squares for sides of up to 11. The sequence is remarkable because the numbers get so large so quickly. At the time of writing this article, no general formula is known and no-one has worked out the exact number for 12 or beyond.

Side	Number of Latin squares
1	1
2	2
3	12
4	576
5	161 280
6	812 851 200
7	61 479 419 904 000
8	108 776 032 459 082 956 800
9	5 524 751 496 156 892 842 531 225 600
10	9 982 437 658 213 039 871 725 064 756 920 320 000
11	776 966 836 171 770 144 107 444 346 734 230 682 311 065 600 000

Table 8

So what does this tell you about the numbers of possible Sudokus?

The side, *s*, of a Sudoku grid must be a square number, so *s* = 1 or 4 or 9 or 16 or … . 105

Apart from the trivial case when *s* = 1, the number of possible Sudokus is always less than the number of Latin squares for the same value of *s*. So the number of possible 4 × 4 Sudokus is less than 576 and the number of 9 × 9 Sudokus is less than the 28-digit number in Table 8.

Given the difficulty of finding a formula for the number of Latin squares, and the fact that the Sudoku grid is by its nature more complicated, it would be surprising if there were an easy general 110
formula for the number of Sudokus of side *s*.

It is not, however, difficult to find the number of 4 × 4 Sudokus, using a similar method to that for Latin squares. Fig. 9 shows one of the 4! = 24 ways of completing the top row.

Because of Sudoku rules the symbol in this cell must be either 3 or 4

Fig. 9

It turns out that there are 16 different ways of filling in the remaining cells while keeping to the Sudoku rules. One of these ways is shown in Fig. 10. 115

Fig. 10

So there are 384 possible 4 × 4 Sudokus. Notice that this is indeed fewer than the 576 Latin squares of the same size.

The method used involved both logic and counting. This mixture of logic and counting has been applied successfully to the 9 × 9 Sudoku by Felgenhauer and Jarvis of the University of Sheffield, but with the difference that, following clever programming, the counting was done by computer. 120
The final outcome was the 22-digit number 6 670 903 752 021 072 936 960.

Investigation 2   What is the smallest number of givens that must be provided if a puzzle is to have a unique solution?

A completed 4×4 Latin square contains the symbols 1, 2, 3 and 4. Look at Fig. 11 as a starting grid for a Latin square puzzle. There are 9 givens and 7 blank cells to be filled in. Only three of the   125
symbols, in this case 1, 2 and 3, appear but you can still find the symbols for the remaining cells easily enough, and there is only one way of doing so.

2	1	3	
1	3		2
3		2	
	2		

Fig. 11

However, if only two of the four symbols, say 1 and 2, appear at the start, there must be more than one way of filling in the grid because the other two symbols, in this case 3 and 4, are interchangeable. The same is true for a 4×4 Sudoku puzzle.   130

So it is certain that for there to be a unique solution to a 4×4 Latin square or Sudoku puzzle, the givens must include at least three distinct symbols. It follows that the number of givens must be at least three.

However, a systematic search shows that there is no unique solution for either a Latin square puzzle or a Sudoku puzzle if there are just three givens.   135

The next case to try is four givens. Fig. 12 shows a Latin square puzzle with four givens and a unique solution.

1		3	
	4		
		1	

Fig. 12

Fig. 13 shows a Sudoku puzzle, also with four givens and a unique solution.

Fig. 13

Interestingly, neither of these is a starting grid for the other type of puzzle.

The situation for the 9×9 Sudoku puzzle is much more complicated because of the increased number   140
of possibilities but the required method is still systematic search, albeit with a well programmed
computer. So far the smallest possible number of givens that anyone has found is 17, and many
examples are now known of such puzzles, including that shown in Fig. 1. At the time of writing
this article no-one has found a Sudoku puzzle with just 16 givens.

Investigation 3   **What is the largest number of givens that can be provided without a 9×9**   145
Sudoku puzzle having a unique solution?

Look at the 4×4 Latin square puzzle in Fig. 14. The givens cover twelve out of the sixteen cells.
However, even with this number of givens, the remaining four cells cannot be filled in uniquely.
There are two possible ways to complete the puzzle. This is because the four missing entries form
an *embedded* Latin square.   150

Fig. 14

Fig. 15 shows the two possible arrangements for the symbols in the four remaining cells. In both
arrangements the four symbols form their own 2×2 Latin square.

Fig. 15

           [Turn over

The four symbols in a 2×2 embedded Latin square do not need to be next to each other but they do need to lie in the four corners of a rectangle. There are three other embedded Latin squares in Fig. 14; one of them is illustrated in Fig. 16.

Fig. 16

The situation would be the same if those twelve givens were provided for the equivalent 4×4 Sudoku puzzle.

If a Latin square puzzle has an embedded Latin square, it will be impossible to solve unless at least one of the symbols in the embedded square is a given. This is also sometimes true for a Sudoku puzzle, as in Fig. 16.

So the largest number of givens that can be provided without a 4×4 Latin square puzzle having a unique solution is at least twelve. In fact it is possible to prove that it is exactly twelve.

In the case of the 9×9 Sudoku puzzle shown in Fig. 17, there are 77 givens and just four cells remain to be filled. However those four form an embedded Latin square and the puzzle does not have a unique solution.

So the largest number of givens that can fail to give a unique solution for a 9×9 Sudoku puzzle is at least 77, and it can be proved that it is exactly 77.

Fig. 17

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4754 (C4) Applications of Advanced Mathematics

Section A

1	$\dfrac{x}{x^2-4}+\dfrac{2}{x+2}=\dfrac{x}{(x-2)(x+2)}+\dfrac{2}{x+2}$ $=\dfrac{x+2(x-2)}{(x+2)(x-2)}$ $=\dfrac{3x-4}{(x+2)(x-2)}$	M1 M1 A1 [3]	combining fractions correctly factorising and cancelling (may be $3x^2+2x-8$)
2	$V=\int_0^1\pi y^2\,dx=\int_0^1\pi(1+e^{2x})\,dx$ $=\pi\left[x+\dfrac{1}{2}e^{2x}\right]_0^1$ $=\pi(1+\dfrac{1}{2}e^2-\dfrac{1}{2})$ $=\dfrac{1}{2}\pi(1+e^2)$ *	M1 B1 M1 E1 [4]	must be π x their y^2 in terms of x $\left[x+\dfrac{1}{2}e^{2x}\right]$ only substituting both x limits in a function of x www
3	$\cos 2\theta=\sin\theta$ $\Rightarrow\quad 1-2\sin^2\theta=\sin\theta$ $\Rightarrow\quad 1-\sin\theta-2\sin^2\theta=0$ $\Rightarrow\quad (1-2\sin\theta)(1+\sin\theta)=0$ $\Rightarrow\quad \sin\theta=\frac12$ or -1 $\Rightarrow\quad \theta=\pi/6,\,5\pi/6,\,3\pi/2$	M1 M1 A1 M1 A1 A2,1,0 [7]	$\cos2\theta=1-2\sin^2\theta$ oe substituted forming quadratic(in one variable) =0 correct quadratic www factorising or solving quadratic $\frac12$, -1 oe www cao penalise extra solutions in the range
4	$\sec\theta=x/2,\ \tan\theta=y/3$ $\sec^2\theta=1+\tan^2\theta$ $\Rightarrow\quad x^2/4=1+y^2/9$ $\Rightarrow\quad x^2/4-y^2/9=1$ * **OR** $x^2/4-y^2/9=4\sec^2\theta/4-9\tan^2\theta/9$ $\qquad\qquad=\sec^2\theta-\tan^2\theta=1$	M1 M1 E1 [3]	$\sec^2\theta=1+\tan^2\theta$ used (oe, e.g. converting to sines and cosines and using $\cos^2\theta+\sin^2\theta=1$) eliminating θ (or x and y) www
5(i)	$dx/du=2u$, $dy/du=6u^2$ $\Rightarrow\quad \dfrac{dy}{dx}=\dfrac{dy/du}{dx/du}=\dfrac{6u^2}{2u}$ $\qquad\qquad=3u$ **OR** $y=2(x-1)^{3/2}$, $dy/dx=3(x-1)^{1/2}=3u$	B1 M1 A1 [3]	both $2u$ and $6u^2$ B1(y=f(x)), M1 differentiation, A1
(ii)	At (5, 16), $u=2$ $\Rightarrow\quad dy/dx=6$	M1 A1 [2]	cao

6(i) $(1+4x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}.4x^2 + \dfrac{(-\frac{1}{2}).(-\frac{3}{2})}{2!}(4x^2)^2 + ...$ $\qquad = 1 - 2x^2 + 6x^4 + ...$ Valid for $-1 < 4x^2 < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$	M1 A1 A1 M1A1 [5]	binomial expansion with $p = -1/2$ $1 - 2x^2 ...$ $+ 6x^4$
(ii) $\dfrac{1-x^2}{\sqrt{1+4x^2}} = (1-x^2)(1-2x^2+6x^4+...)$ $= 1 - 2x^2 + 6x^4 - x^2 + 2x^4 + ...$ $= 1 - 3x^2 + 8x^4 + ...$	M1 A1 A1 [3]	substitituting their $1 - 2x^2 + 6x^4 + ...$ and expanding ft their expansion (of three terms) cao
7 $\sqrt{3}\sin x - \cos x = R\sin(x-\alpha)$ $\qquad = R(\sin x\cos\alpha - \cos x\sin\alpha)$ $\Rightarrow \quad \sqrt{3} = R\cos\alpha,\ 1 = R\sin\alpha$ $\Rightarrow \quad R^2 = 3+1 = 4 \Rightarrow R = 2$ $\qquad \tan\alpha = 1/\sqrt{3}$ $\quad \Rightarrow \alpha = \pi/6$ $\Rightarrow \quad y = 2\sin(x-\pi/6)$ Max when $x - \pi/6 = \pi/2 \Rightarrow x = \pi/6 + \pi/2 = 2\pi/3$ max value $y = 2$ So maximum is $(2\pi/3, 2)$	M1 B1 M1 A1 B1 B1 [6]	correct pairs soi $R = 2$ ft cao www cao ft their R SC B1 $(2, 2\pi/3)$ no working

Section B

8(i) At A: $3 \times 0 + 2 \times 0 + 20 \times (-15) + 300 = 0$ At B: $3 \times 100 + 2 \times 0 + 20 \times (-30) + 300 = 0$ At C: $3 \times 0 + 2 \times 100 + 20 \times (-25) + 300 = 0$ So ABC has equation $3x + 2y + 20z + 300 = 0$	M1 A2,1,0 [3]	substituting co-ords into equation of plane… for ABC **OR** using two vectors in the plane form vector product M1A1 then $3x + 2y + 20z = c = -300$ A1 **OR** using vector equation of plane M1,elim both parameters M1, A1
(ii) $\overrightarrow{DE} = \begin{pmatrix} 100 \\ 0 \\ -10 \end{pmatrix}$ $\overrightarrow{DF} = \begin{pmatrix} 0 \\ 100 \\ 5 \end{pmatrix}$ $\begin{pmatrix} 100 \\ 0 \\ -10 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 20 \end{pmatrix} = 100 \times 2 + 0 \times -1 + -10 \times 20 = 200 - 200 = 0$ $\begin{pmatrix} 0 \\ 100 \\ 5 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 20 \end{pmatrix} = 0 \times 2 + 100 \times -1 + 5 \times 20 = -100 + 100 = 0$ Equation of plane is $2x - y + 20z = c$ At D (say) $c = 20 \times -40 = -800$ So equation is $2x - y + 20z + 800 = 0$	B1B1 B1 B1 M1 A1 [6]	 need evaluation need evaluation
(iii) Angle is θ, where $\cos\theta = \dfrac{\begin{pmatrix} 2 \\ -1 \\ 20 \end{pmatrix} . \begin{pmatrix} 3 \\ 2 \\ 20 \end{pmatrix}}{\sqrt{2^2 + (-1)^2 + 20^2} \sqrt{3^2 + 2^2 + 20^2}} = \dfrac{404}{\sqrt{405}\sqrt{413}}$ $\Rightarrow$ $\theta = 8.95°$	M1 A1 A1 A1cao [4]	formula with correct vectors top bottom (or 0.156 radians)
(iv) RS: $\mathbf{r} = \begin{pmatrix} 15 \\ 34 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 20 \end{pmatrix}$ $= \begin{pmatrix} 15 + 3\lambda \\ 34 + 2\lambda \\ 20\lambda \end{pmatrix}$ $\Rightarrow$ $3(15 + 3\lambda) + 2(34 + 2\lambda) + 20.20\lambda + 300 = 0$ $\Rightarrow$ $45 + 9\lambda + 68 + 4\lambda + 400\lambda + 300 = 0$ $\Rightarrow$ $413 + 413\lambda = 0$ $\Rightarrow$ $\lambda = -1$ so S is $(12, 32, -20)$	B1 B1 M1 A1 A1 [5]	$\begin{pmatrix} 15 \\ 34 \\ 0 \end{pmatrix} + ...$ $... + \lambda \begin{pmatrix} 3 \\ 2 \\ 20 \end{pmatrix}$ solving with plane $\lambda = -1$ cao

9(i) $v = \int 10e^{-\frac{1}{2}t}dt$	M1	separate variables and intend to integrate
$= -20e^{-\frac{1}{2}t} + c$	A1	$-20e^{-\frac{1}{2}t}$
when $t = 0$, $v = 0$	M1	finding c
$\Rightarrow \quad 0 = -20 + c$		
$\Rightarrow \quad c = 20$	A1	cao
so $v = 20 - 20e^{-\frac{1}{2}t}$	[4]	
(ii) As $t \to \infty$ $e^{-1/2\,t} \to 0$	M1	
$\Rightarrow \quad v \to 20$		
So long term speed is 20 m s^{-1}	A1	ft (for their $c>0$, found)
	[2]	
(iii) $\dfrac{1}{(w-4)(w+5)} = \dfrac{A}{w-4} + \dfrac{B}{w+5}$	M1	
$= \dfrac{A(w+5) + B(w-4)}{(w-4)(w+5)}$		
$\Rightarrow \quad 1 \equiv A(w+5) + B(w-4)$		
$w = 4$: $1 = 9A \Rightarrow A = 1/9$	M1	cover up, substitution or equating coeffs
$w = -5$: $1 = -9B \Rightarrow B = -1/9$	A1	1/9
$\Rightarrow \quad \dfrac{1}{(w-4)(w+5)} = \dfrac{1/9}{w-4} - \dfrac{1/9}{w+5}$	A1	$-1/9$
$= \dfrac{1}{9(w-4)} - \dfrac{1}{9(w+5)}$	[4]	
(iv) $\dfrac{dw}{dt} = -\dfrac{1}{2}(w-4)(w+5)$		
$\Rightarrow \quad \int \dfrac{dw}{(w-4)(w+5)} = \int -\dfrac{1}{2}dt$	M1	separating variables
$\Rightarrow \int [\dfrac{1}{9(w-4)} - \dfrac{1}{9(w+5)}]dw = \int -\dfrac{1}{2}dt$	M1	substituting their partial fractions
$\Rightarrow \dfrac{1}{9}\ln(w-4) - \dfrac{1}{9}\ln(w+5) = -\dfrac{1}{2}t + c$	A1ft	integrating correctly (condone absence of c)
$\Rightarrow \dfrac{1}{9}\ln\dfrac{w-4}{w+5} = -\dfrac{1}{2}t + c$		
When $t = 0$, $w = 10 \Rightarrow c = \dfrac{1}{9}\ln\dfrac{6}{15} = \dfrac{1}{9}\ln\dfrac{2}{5}$	M1	correctly evaluating c (at any stage)
$\Rightarrow \ln\dfrac{w-4}{w+5} = -\dfrac{9}{2}t + \ln\dfrac{2}{5}$	M1	combining lns (at any stage)
$\Rightarrow \dfrac{w-4}{w+5} = e^{-\frac{9}{2}t + \ln\frac{2}{5}} = \dfrac{2}{5}e^{-\frac{9}{2}t} = 0.4e^{-4.5t}$ *	E1	www
	[6]	
(v) As $t \to \infty$ $e^{-4.5\,t} \to 0$	M1	
$\Rightarrow \quad w - 4 \to 0$	A1	
So long term speed is 4 m s^{-1}.	[2]	

Comprehension

1. (i)

2	1	3
3	**2**	**1**
1	**3**	**2**

B1
cao

   (ii)

2	3	**1**
3	**1**	**2**
1	2	**3**

B1
cao

2. Dividing the grid up into four 2 x 2 blocks gives

1	2	3	4
3	1	4	2
2	4	1	3
4	3	2	1

Lines drawn on diagram or reference to 2 x 2 blocks.   **M1**

One (or more) block does not contain all 4 of the symbols 1, 2, 3 and 4. oe.   **E1**

3.

1	2	3	4
4	**3**	**1**	**2**
2	**1**	**4**	**3**
3	**4**	**2**	**1**

Many possible answers   Row 2 correct   **B1**

Rest correct   **B1**

4.   **Either**   **Or**   **B2**

4	2	3	1
		2	4
		4	2
2	4	1	3

4	2	3	1
		2	4
		4	2
2	4	1	3

5.   In the top row there are 9 ways of allocating a symbol to the left cell, then 8 for the next, 7 for the next and so on down to 1 for the right cell, giving   **M1**

$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 9!$ ways.   **E1**

So there must be $9! \times$ the number of ways of completing the rest of the puzzle.

6.

(i)

Block side length, b	Sudoku, $s \times s$	M
1	1×1	-
2	4×4	12
3	9×9	**77**
4	16×16	**252**
5	**25×25**	**621**

                      25×25   **B1**

                 77, 252 and 621   **B1**

(ii) $M = b^4 - 4$ b^4 **B1**

                                       - 4   **B1**

7.

(i)   There are neither 3s nor 5s among the givens.   **M1**

            So they are interchangeable and therefore there is no unique solution   **E1**

(ii)   The missing symbols form a 3×3 embedded Latin square.   **M1**

            There is not a unique arrangement of the numbers 1, 2 and 3 in this square.   **E1**

                                                                              [18]

4754 Applications of Advanced Mathematics (C4)

General Comments
Many candidates found this paper was easier than that of last summer but more difficult than that in January this year. Section A included sufficient questions to challenge the good candidates but also many questions that were accessible to weaker candidates. Thus, there were very few low marks overall. The Comprehension scored highly with a number of straightforward numerical questions. Few, however, scored full marks here. Although candidates often appeared to have some understanding of the answers for the worded questions, they often did not give sufficient information to achieve the marks involved.
The presentation was generally better than in previous years.

Comments on Individual Questions

      Paper A

Section A

1)   Many candidates scored only one mark from the three available here when attempting to add algebraic fractions. The majority of candidates knew and used the correct method for adding two fractions. However, only the minority factorised and cancelled their expression. In most cases the denominator $(x^2-4)(x-2)$ was used and the numerator $3x^2+2x-8$ was found but then candidates failed to continue. Only a small number realised that the simplest denominator was $(x+2)(x-2)$.

2)   Most candidates correctly found the volume of revolution. Errors included failing to substitute the lower limit, failing to include the π in the term for the second limit and errors in integrating e^{2x}.

3)   There were many completely correct solutions to this question. Some started badly by making an incorrect substitution for $\cos2\theta$. $2\sin\theta\cos\theta$ and $1-\sin^2\theta$ were commonly seen. Those that substituted correctly were usually successful provided that they realised that their equation was a quadratic equation to be solved. Almost all candidates gave their answers in the required exact form in terms of π although some missed the solution $5\pi/6$.

4)   There were some good, clear concise answers to this question, but they were in the minority. There was some very poor setting out of a logical argument. Candidates frequently failed to 'show that' sufficiently to achieve the mark for establishing the result. Common errors included $2\sec\theta = 1/2\cos\theta$ and $(3\sec\theta)^2 = 3\sec^2\theta$. Candidates need to improve their presentation of coherent arguments in 'proof' questions.

5)   Almost all candidates obtained full marks on this question. Using the chain rule was the most common approach. The most frequent error was to substitute $u=5$ in part (ii). Occasionally a final answer of ±6 was seen.

6)   Most candidates used the Binomial expansion correctly in (i). Occasionally the powers of ½ or -1 were used instead of -1/2. There were a few numerical errors in evaluating the third term. The commonest error, however, in (i) was the validity. This was often omitted and frequently wrong. Errors included, $|4x|{<}1$, $|4x^2|{\leq}1$, $-1/2{\leq}x{\leq}1/2$ and $\sqrt{-1/4}{<}x{<}\sqrt{1/4}$. It was disappointing, although not penalised, to find so many candidates failing to simplify $\sqrt{1/4}$ to 1/2.

Most candidates correctly used their expression from (i) in (ii) although weaker candidates failed to recognise the usefulness or relevance of part (i). A few tried to expand $(1{-}x^2)$ in (ii) using the Binomial expansion again.

7)   Many candidates approached the solution to this question by learning the process rather than from first principles often leading to $\tan\alpha{=}\sqrt{3}$ instead of $1/\sqrt{3}$. Sign errors, commonly $-1{=}R\sin\alpha$ were seen and again, disappointingly, $\sqrt{4}$ left unresolved was often seen for R- although most candidates did obtain $R{=}2$. The final part was usually either completely correct or omitted.

Section B

8)   (i)   A wide variety of methods were seen. These included substituting the co-ordinates of the points in the equation of the plane, using a vector product and a combination of substituting points and the scalar product of vectors in the plane with the normal vector. Finding the Cartesian equation of the plane from the vector equation was also often used. Many candidates (even the most able students) failed to realise that more than one point substituted was needed and also failed to realise that one scalar product, or two, alone is insufficient to verify the Cartesian equation of the plane.

      (ii)   This was usually successful and full working was shown. The last part was sometimes again found from the vector equation of the plane. Some gave the final part as $2x{-}y{+}20z{=}0$ or $2x{-}y{+}20z{=}800$.

      (iii) The scalar product was usually correct provided the correct vectors were used although there were numerical errors.

      (iv) The correct vector equation was usually given although '$r =$' was often omitted. The final answer was also often found but some failed to realise that they needed to substitute into the equation of the plane. A frequent error was '$413{+}413\lambda{=}0,\ \lambda{=}1$'.

9)   (i)   Most candidates separated the variables and attempted to integrate. Most did not include a constant. Common incorrect answers being

$v{=}{\pm}20e^{-\frac{1}{2}t}, 10e^{-\frac{1}{2}t}$ and $5e^{-\frac{1}{2}t}$.

      (ii)   For good candidates who had found 'c' in (i), (ii) was usually successful. Many failed to realise that $e^{-\frac{1}{2}t}\to 0$, choosing ∞ or 1 instead. Some substituted a large value for t which was condoned in (ii) but not in (v).

      (iii) The partial fractions were almost always correct.

      (iv) Some candidates made a poor start here, failing to separate the variables and only working with their partial fractions from (iii). For those that separated the variables correctly, their initial integration was usually correct and they combined their logarithms correctly but for those that included a constant it was rarely found successfully. The correct final answer was given and often appeared without derivation. For those evaluating a constant a **frequent error**, as in the past, was to separate exp(-4.5t+c) as exp(-4.5t) +exp c.

      (v)   Few found that $w{\to}4$ as $t{\to}\infty$. $w{\to}0$ was fairly common.

Paper B - The Comprehension

1)   The tables were almost always filled in correctly.

2)   Many candidates failed to make it clear that they were discussing 2x2 blocks in their explanations. Some referred to 3x3 blocks or blocks in general or said that a Sudoku had to be 9x9. For those that did indicate 2x2 blocks, they usually gave the required reason for the Latin Square not being a Sudoku puzzle which is because there were repeated entries in the individual blocks.

3)   This table was usually completed correctly although some gave Fig. 10.

4)   This was usually correct.

5)   This was not well done. Some said that because there were 9 squares it must be 9! rather than explaining this was the number of ways to fill in a row of 9. Some said if 3x3 was 3! And 4x4 was 4! then this must be 9!. Others failed to explain that the 9! would need to be multiplied by the number of ways for the rest of the grid to get the total number.

6)   This was often completely correct. Most found 25x25 correctly but there were errors in the second column. In (ii) common errors were $M=b^4-b^2$ and using s instead of b.

7)   (i)   Those that realised that all the 3s and 5s were missing from the givens were usually successful. Others missed the point and said there was no unique solution because there were less than 77 givens.

(ii)   There were many confused explanations here often referring, again, to the total number of givens rather than discussions about the numbers 1,2 and 3 forming an embedded square with 12 different solutions possible.